3D INTERPOLATION USING HANKEL TENSOR COMPLETION BY ORTHOGONAL MATCHING PURSUIT

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Introduction. Seismic data are often sparsely or irregularly sampled along one or more spatial axes. Irregular sampling can produce artifacts in seismic imaging results, thus multidimensional interpolation of seismic data is often a key processing step in exploration seismology. Many solution methods have appeared in the literature: for instance in Spitz (1991), Sacchi (2000) it was proposed to perform seismic trace interpolation that handles spatially aliased events using the linear predictors to estimate the missing traces through linear filters in the f-x(y) domain.

Conversely, multidimensional Fourier reconstruction methods exploit a different data representation domain, while still assuming that seismic data consists of a superposition of plane waves: statistical sparsity is assumed in order to retrieve a model that consists of a few dominant wavenumbers representing the observations (Liu, 2004; Sacchi, 2010). In Xu (2004, 2005) and Sacchi (2000) the so-called Antileakage Fourier Transform (ALFT) algorithm was proposed: it tries to resolve, or at least attenuate, the spectral leakage of the irregular Fourier Transform by taking advantage of the compressive sensing framework (data are assumed to be sparse in the frequency-wavenumber domain).

In more recent years, methods based on alternative local transformations, such as curvelet and seislet transforms (Herrmann, 2010; Fomel, 2010), have also been proposed.

Recently, new interpolators have been developed recasting the interpolation problem to a compressive sensing matrix completion problem (Candes, 2009; Yang, 2013; Herrmann, 2014). A matrix with randomly missing entries can be completed by solving the rank minimization problem under the low-rank assumption of the underlying solution, that is in opposition to the fact that the subsampling operator tends to increase the rank of the matrix. Trickett (2010) proposed to apply a Cadzow filtering to solve the trace interpolation problem. The Cadzow algorithm replaces the block Hankel matrix of the incomplete and noisy multidimensional observations: it allows to recover the complete data volume by its low rank approximation and successive averaging along the anti-diagonals of the rank-reduced Hankel matrix.

In recent literature, the low-rank tensor completion technique has become widely used in many areas such as computer vision, signal processing and seismic data analysis. This approach is based on a high order generalization of the low-rank matrix completion problem, and can be approximated by convex relaxation which minimizes the nuclear norm instead of the rank of the tensor. In Trickett (2013) a generalization of Hankel matrices is applied to the tensor to perform an efficient completion.

In order to extend these approaches from matrix to tensor completion, we make use of extended notions on the concept of rank borrowed from linear algebra in conjunction with the high dimensional tensor theory.

Tensor algebra is a mathematical framework that generalizes the concepts of vectors and matrices to higher dimensions. These tools are widely used to address problems of missing data in biomedical signal processing, computer vision, image processing, communication and seismic data processing.

The High Order Singular Value Decomposition (HOSVD; Kolda, 2009) is used in many tensor rank-reduction algorithms. The HOSVD technique can be viewed as a generalization of the classical SVD for tensors. The computation of the HOSVD (De Lathauwer, 2000; Bergqvist, 2010) requires performing SVD as many times as the order of the tensor, over all the possible matrix representations of the tensor. Due to this fact the HOSVD can be expensive in terms of computational time.

In tensor completion, the goal is to fill missing entries of a partially known tensor, under a low-rank condition. Many algorithms have been proposed in literature (Da Silva, 2013; Kreimer,

2012) to solve the low-rank tensor completion problem with global optimization methods.

In this work, we want to introduce a new greedy algorithm, called Orthogonal Rank-One Tensor Pursuit (OR1TP), that extends the Orthogonal Matching Pursuit (OMP) algorithm (Pati, 1993; Davis, 1994) that works in multidimensional tensor space and finds a low rank approximation of a tensor. We present a novel seismic interpolation algorithm based on the rank reduction technique, that solves the low-rank Hankel tensor completion problem in the frequency domain and that is capable to fill missing seismic traces in 3D dataset. Finally, we show experimental results obtained on synthetic data to prove the effectiveness of our method.

Theory. A tensor is the generalization of vectors, matrices to higher dimensions. Let $Y \in C^{n_1 \times n_2 \times \ldots \times n_N}$ be a *N*-th order tensor, while $\Omega \subset \{1,\ldots,n_1\} \times \{1,\ldots,n_2\} \times \ldots \times \{1,\ldots,n_N\}$ denotes the indexes of the observed entries of *Y*. The order *N* of a tensor is the number of dimensions, also known as ways or modes.

A second-order tensor is a matrix and a first-order tensor is a vector. Let P_Ω be the projector onto the space such that the indexes $(i_1, i_2, \ldots, i_N) \in \Omega$.

It is well known that the rank of a matrix coincides with its column and row rank. The rank of a tensor is a much trickier concept. The rank of a tensor *Y* is equal to the minimum number of rank-one tensors that yield *Y* in a linear combination.

Any tensor $Y \in C^{n_1 \times n_2 \times \ldots \times n_N}$ can be written as a linear combination of rank-one tensors, that is

$$Y = \sum_{i=1}^{R} \theta_i \,(u_1^{(i)} \circ u_2^{(i)} \circ \ldots \circ u_N^{(i)}) = \sum_{i=1}^{R} \theta_i T_i = T(\theta)$$

where $\lambda^{(i)} \in \boldsymbol{R}$, $u_j^{(i)} \in C^{n_j}$ are rank-one tensors such that $\langle u_j^{(i)}, u_j^{(k)} \rangle = 0 \; \forall i \neq k$ and *r* is the rank of the tensor *Y*.

Rank-One tensor decomposition, also called Canonical Polyadic Decomposition (CPD; Zhang, 2001), gives a compact representation of the underlying structure of the tensor, revealing when the tensor-data can be modeled as lying close to a low dimensional subspace.

The low-rank tensor completion problem aims to minimize the zero-norm of θ subject to the equality constraint $P_\Omega(T(\theta)) = P_\Omega(Y)$ as follows:

$$\min_{\theta} \|\theta\|_0 \qquad s.t.\; P_\Omega(T(\theta)) = P_\Omega(Y) \qquad (P0)$$

Unfortunately, direct (P0) norm minimization problem is NP-hard (nondeterministic polynomial time hard

problem) because it is equivalent to a subset selection problem, which is itself an NP-hard combinatorial optimization problem (Donoho, 2004).

Orthogonal Rank-One Tensor Pursuit. We can relax the original problem by rewriting (P0) as follows:

$$\min_{\theta} \left\|P_\Omega(T(\theta)) - P_\Omega(Y)\right\|^2 \qquad s.t.\; \|\theta\|_0 \leq r \qquad (G0)$$

The main idea is to extend the Orthogonal Matching Pursuit (OMP) procedure (Pati, 1993; Davis, 1994) from the vector field to the tensor field, introducing a novel greedy algorithm called Orthogonal Rank-One Tensor Pursuit (***OR1TP***) that solve (G0) problem in an efficient way. Following the idea proposed by (Wang, 2014), at each iteration a rank-one basis tensor is generated by the vectors that approximate the residual tensor of the data. After that, the weights of the current rank-one tensor bases are updated by performing an orthogonal projection of the observation tensor on their spanning subspace. The most time consuming step of this method is the computation of the tensor $T = u_1 \circ u_2 \circ \ldots \circ u_N$ that best approximate the residual $R_k = P_\Omega(Y) - \sum_{i=1}^{k-1}(T_i)_\Omega$. This step requires to solve the maximization problem:

$$\max_T \{ \sum_{i_1, i_2, \ldots, i_N} T(i_1, i_2, \ldots, i_N) R_k(i_1, i_2, \ldots, i_N) \;\; s.t. \;\; rank(T) = 1, \quad \|T\| = 1 \}$$

In Zhang (2001) iterative methods, based on the Multilinear Rayleigh Quotients, are suggested to find a rank-one approximation of $R_k R_k$ in an efficient way. Differently from the classical Orthogonal Matching Pursuit algorithm, that requires the storage of the entire vectors bases, the proposed algorithm estimates a basis tensor only once, allowing an efficient memory management.

Orthogonal Rank-One Tensor Pursuit (OR1TP)

- **Input:** Y_Ω and a tolerance parameter ϵ
- **Initialize:** $X_0 = 0$, $\theta^k = 0$, $k = 1$
- Repeat
  - Find a pair of singular tensors (x_k, y_k, w_k, z_k) of the observed residual $R_k = Y_\Omega - X_{k-1}$, set $T_k(i,j,l,k) = x(i)\, y(j)\, w(l)\, z(k)$
  - Solve $\widehat{T}_k \theta^k = \hat{y}$, $\widehat{m}_I$ reshape of $(T_k)_\Omega$, $\hat{y}$ reshape of Y_Ω
  - Set $X_k = \sum \theta_i^k (T_i)_\Omega$
- Until $\|R_k\| < \epsilon$
- **Output:** $\hat{Y} = \sum \theta_i^k T_i$

Tab. 1. Orthogonal Rank-One Tensor Pursuit (OR1TP) solve the (OMP) by an orthogonal matching pursuit type greedy algorithm using rank-one tensors as the basis. Given a tensor Y_Ω with missing values, finds a tensor $\hat{Y}$ which is a low-rank approximation of the observed tensor.

Application to data interpolation problem. Recently, many trace interpolators based on low-rank tensor completion have been proposed. Our interpolation method considers 3D spatial data in frequency/space domain. We consider the case where seismic traces are attributed to a regular 3D grid through binning. In the case that more than one trace is assigned to a bin, we average them to retain only one observation in each bin. In real situations, mapping seismic traces from an irregular to a regular grid through the binning process, leads to a highly sparse volume with missing traces randomly disposed.

Given a 3D seismic dataset in time-space domain, in order to recover missing traces, we first perform a 1D discrete Fourier transform on the time axis.

A symmetric tensor is a tensor that is invariant under a permutation of its vector arguments $T(i_1, \ldots, i_N) = T(\sigma_1, \ldots, \sigma_N)$ for any permutation σ. Symmetric tensors form a singular important class of tensors. Hankel tensors are symmetric tensors that originate from applications such as signal processing. Due to the fact that Hankel tensors are symmetric and then well structured, the low-rank assumption is enhanced by the geometrical structure of the data.

To recover the missing samples, each frequency slice of the data in frequency-space domain is rearranged in a 4D Hankel tensor. This tensor can be obtained with an appropriate transformation as described in Trickett (2010). Given a raw 2D frequency slice $\in \boldsymbol{C}^{m \times n}$, to form the 4D Hankel tensor $\boldsymbol{T} \in \boldsymbol{C}^{p \times q \times r \times s}$, we apply the following:

$$T(i, j, k, l) = S(i + j - 1, k + l - 1)$$

where the four tensor directions have length, respectively, $p = \frac{m}{2} + 1$, $q = \frac{m+1}{2}$, $r = \frac{n}{2} + 1$ and $t = \frac{n+1}{2}$, with $i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\},\ k \in \{1, \ldots, r\}$ and $l \in \{1, \ldots, s\}$.

Once the tensor *T* is obtained, the low-rank tensor completion is performed by the OR1TP algorithm and the interpolated frequency slice is obtained applying the inverse transform.

The interpolation procedure described above is summarized in Tab. 2.

- Take the Discrete Fourier Transform (DFT) of each trace in the grid
- For each frequency within the signal band
  - Form a complex-value Hankel tensor *T*
  - Perform tensor completion on *T* by OR1TP algorithm
  - Recover the interpolated frequency slice from the completed tensor
- Take the Inverse Discrete Fourier Transform (IDFT) of each trace.

Tab. 2. Interpolation procedure by low-rank Hankel tensor completion in frequency domain performed by the OR1TP greedy algorithm.

Examples. We chose to test the proposed approach with a portion of the synthetic SEG Advanced Modeling Program (SEAM) 3D dataset to prove the effectiveness of our method. We obtained an irregularly sampled dataset from the (regularly sampled) reference data by removing a randomly selected subset of traces. Two different parameter sets have been tested: in the first experiment (Fig. 1), each 2D frequency slice is of size 7 × 7 samples, that rearranged in a Hankel tensor becomes a 4D volume of size 4 × 4 × 4 × 4. Due to the fact that the maximum rank of our 4D Hankel tensor is $n^3 = 64$, we recovered each tensor by approximating it with a tensor having maximum rank equal $\frac{n^3}{2} = 32$. In the second experiment, the frequency slice is a window of size 5 × 5 samples that, once mapped to a Hankel tensor, becomes a multidimensional array of order 4 and size 3 × 3 × 3 × 3. Similarly to the experiment previously described, the maximum rank was chosen equal to $\left\lceil \frac{n^3}{2} \right\rceil = 14$.

To better highlight the interpolation capability of the proposed approach, both experiments have being carried out by eliminating half of the traces in the original 3D dataset.

The computation of the rank-one tensor that approximates the residual tensor at each iteration of the OR1TP is performed through the generalized iterative Rayleigh quotient method (Lathauwer, 2000), choosing as stopping criteria either maximum number of iterations *MaxIter* = 20 and tolerance *tol* =1e-3.

Fig. 1 shows the results obtained by OR1TP greedy algorithm with frequency slices of size 7 × 7 samples over a portion of the data containing linear events.

Fig. 2 shows the results obtained by filling the missing traces with OR1TP algorithm. In this case, the frequency slice size is equal to 5 × 5 samples. In this portion of data there is the presence of hyperbolic events with high curvature that does not influence the results of the reconstruction.

To evaluate quantitatively the performance of our interpolation algorithm, we chose to vary the percentage of randomly zeroed traces from 20% to 60% for fixed instances (5×5 and

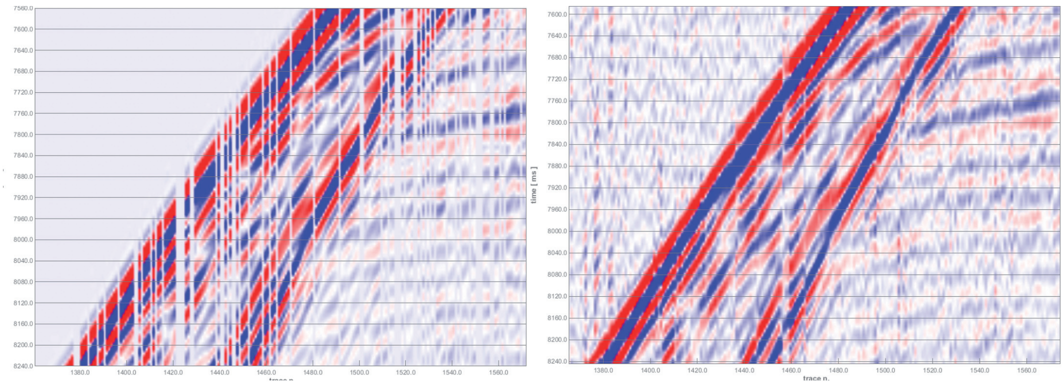

Fig. 1 - Results obtained on a portion of data with presence of linear events. Subset of synthetic data with 50% of missing traces (left), interpolation result obtained by OR1TP algorithm (right).

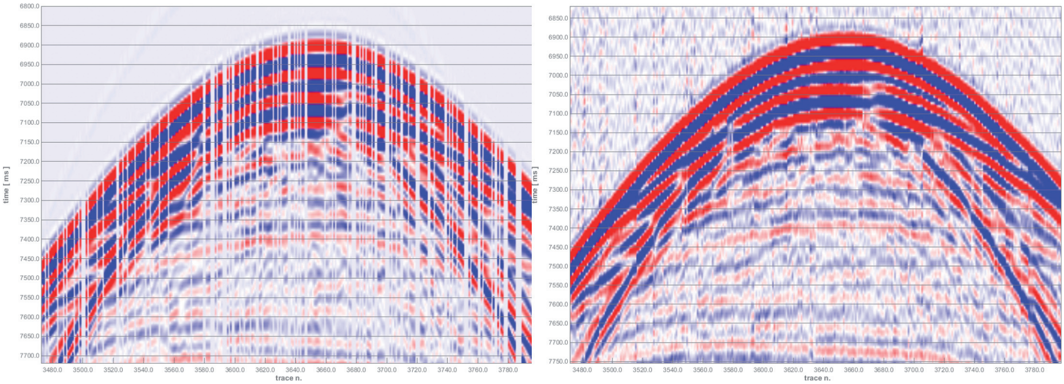

Fig. 2 - Results obtained on a portion of data with presence of hyperbolic events. Subset of synthetic data with 50% of missing traces (left), interpolation result obtained by OR1TP algorithm (right).

7×7 frequency slice size), and to calculate the Signal to Noise Ratio of the recovered data in time domain. Results are showed in Fig. 3. As it was easy to guess, the reconstruction quality decreases as the percentage of censored traces increases, maintaining acceptable values even with high number of missing traces.

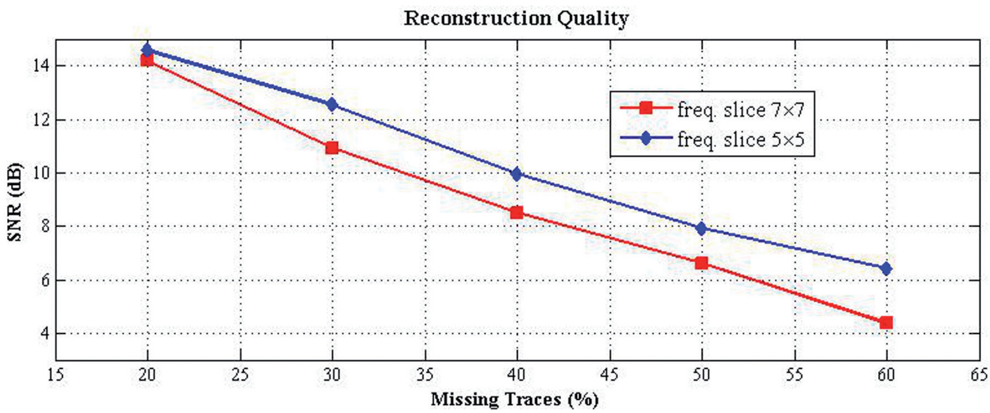

Fig. 3 - Quality reconstruction obtained for 5×5 and 7×7 frequency slice size, varying the number of random censored traces of the 3D data. The quality measure is the Signal to Noise Ratio (SNR) in dB.

The parameters which are required by our interpolation algorithm are the spatial window size and the maximum rank of the tensor. The spatial window size is a crucial parameter that can influence significantly the computational time since it affects the dimensions of the tensors to be completed. There are no known methods in the literature to estimate the optimal rank of each tensor. However, this fact does not influence significantly the reconstruction quality as long as the maximum rank is set to a value comparable to the largest fraction of the energy recalled by the approximating tensor. In all our experiments, it was enough to choose the maximum rank of the approximating tensor equal to half of the maximum rank of the tensor of observations.

It is important to notice that the curvature of the events has not affected the quality of the reconstruction, thus demonstrating empirically that the proposed method turns out to be independent to the shape of the events. This appears to be an important advantage with respect to Spitz-like methods or algorithms based on local transform like seislet or curvelet, requiring proper geometrical assumptions**.**

Similarly to all low-rank tensor completion methods, the interpolation algorithm proposed in this paper cannot handle spatially aliased data, but only dense data in a regular grid with irregularly missing traces.

Conclusions. In this work we have extended the greedy Orthogonal Matching Pursuit algorithm to the multidimensional case, making making it applicable to low-rank tensor completion problem.

We have introduced a new method for reconstructing and interpolating missing traces in 3D datasets.

The algorithm operates in frequency-space domain, solving, for each temporal frequency, instances of the low-rank Hankel completion problem by the proposed OR1TP greedy algorithm. At each iteration this algorithm searches for a rank-one tensor approximation of the residual, making the method efficient in terms of time and solution quality.

Synthetic data extracted from the synthetic SEG Advanced Modeling Program (SEAM) dataset, was used to prove the ability of the proposed method to recover missing traces in a 3D irregularly sampled dataset. Since there are no assumptions about the data, the algorithm turns out to be robust with respect to the shape of the events, obtaining good results both in case of linear and curved events.

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